

Equiangular subspaces in Euclidean spaces

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Abstract

A set of lines through the origin is called equiangular if every pair of lines defines the same angle, and the maximum size of an equiangular set of lines in \mathbb{R}^n was studied extensively for the last 70 years. In this paper, we study analogous questions for k -dimensional subspaces. We discuss natural ways of defining the angle between k -dimensional subspaces and correspondingly study the maximum size of an equiangular set of k -dimensional subspaces in \mathbb{R}^n . Our bounds extend and improve a result of Blokhuis.

1 Introduction

A set of lines passing through the origin is called equiangular if every pair of lines makes the same angle. The question of determining the maximum size $N(n)$ of a set of equiangular lines in \mathbb{R}^n has a long history going back 70 years. It is considered to be one of the founding problems of algebraic graph theory, see [2, 3, 7, 11, 13, 15] and references for more information. It is known that $N(n)$ grows quadratically with n . The upper bound

$$N(n) \leq \binom{n+1}{2} \quad (1)$$

was proved by Gerzon (see [13]) and de Caen [5] gave a construction showing

$$N(n) \geq \frac{2}{9}(n+1)^2 \quad (2)$$

for all n of the form $3 \cdot 2^{2t-1} - 1$ where $t \in \mathbb{N}$.

It is therefore natural and interesting to study analogous questions for k -dimensional subspaces. Note that we will always represent a k -dimensional subspace in \mathbb{R}^n by an $n \times k$ matrix $U = (u_1, \dots, u_k)$ where u_1, \dots, u_k is an orthonormal basis spanning the subspace, with the implicit understanding that any definition involving U will not depend on the particular choice of orthonormal basis. We define the *Grassmannian* $\text{Gr}(k, n)$ to be the set of all k -dimensional subspaces of \mathbb{R}^n . For any $U, V \in \text{Gr}(k, n)$, it is well known that there are k principle angles $0 \leq \theta_1 \leq \dots \leq \theta_k \leq \pi/2$ between U and V , defined by $\theta_i = \arccos(\sqrt{\lambda_i})$ where $1 \geq \lambda_1 \geq \dots \geq \lambda_k \geq 0$ are the eigenvalues of $V^\top U U^\top V$. They completely characterize the relative position of U to V , in the sense that if U', V' have the same principle angles as U, V , then there exists an orthogonal matrix Q such that $U' = QU$ and $V' = QV$, see [16, Theorem 3].

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One can consider equiangular sets of subspaces with respect to θ_i for any fixed $1 \leq i \leq k$. More generally, one could take any function $d = d(\theta_1, \dots, \theta_k)$ of the principle angles and study $N_\alpha^d(k, n)$, the maximum size of an equiangular set $H \subseteq \text{Gr}(k, n)$ such that $d(U, V) = \alpha$ for all $U \neq V \in H$, as well as $N^d(k, n) = \max_\alpha N_\alpha^d(k, n)$. We call a function $d : \text{Gr}(k, n)^2 \rightarrow \mathbb{R}$ an *angle distance* if $d(U, V) \in \{\theta_1(U, V), \dots, \theta_k(U, V)\}$ for all $U, V \in \text{Gr}(k, n)$. If d also satisfies $d(U, V) = 0$ iff $U = V$ then we call d a *proper angle distance*. In section 2 we give examples of angle distances and prove a general upper bound on $N_\alpha^d(k, n)$ for any angle distance d and $\alpha > 0$, in particular improving and extending a result of Blokhuis [4] who studied the case $d = \theta_1$ and $k = 2$. Based on equiangular lines, we also give a lower bound construction of k -dimensional subspaces that are equiangular for any proper angle distance. We therefore conclude that for k fixed and any proper angle distance d , $N^d(k, n) = \Theta(n^{2k})$ as $n \rightarrow \infty$.

2 Angle distances

When trying to define the angle between two subspaces $U, V \in \text{Gr}(k, n)$, one natural idea is to just take the minimum angle between any pair of vectors $u \in U, v \in V$. Since minimizing $\arccos(\langle u, v \rangle)$ is equivalent to maximizing $\langle u, v \rangle$, it is not hard to see that this idea gives exactly the first principle angle $\theta_1 = \theta_1(U, V)$. This angle distance was first considered by Dixmier [9]. In [4], Blokhuis considered equidistant planes with respect to θ_1 and proved that

$$N_\alpha^{\theta_1}(2, n) \leq \binom{2n+3}{4} \quad (3)$$

provided that the common angle $\alpha > 0$. This condition is necessary, since $\theta_1(U, V) = 0$ iff U and V share a nontrivial subspace, and so we could take infinitely many planes all sharing a fixed line, showing that $N_0^{\theta_1}(2, n) = \infty$. This is a troublesome property of θ_1 , because it shows that θ_1 is not a metric and also that θ_1 does not appeal to elementary geometric intuition. Indeed, consider a pair of planes U, V in \mathbb{R}^3 . They will always share a line and hence will have $\theta_1(U, V) = 0$. However, one would intuitively ascribe the angle between them to be $\theta_2(U, V)$.

In view of this, it makes sense to define the minimum non-zero angle $\theta_F(U, V) = \min\{\theta_i(U, V) : \theta_i(U, V) > 0\}$. θ_F was first considered by Friedrichs [10] and it is a proper angle distance. Deutsch [8] gives applications of θ_1 and θ_F to the rate of convergence of the method of cyclic projections, existence and uniqueness of abstract splines, and the product of operators with closed range. Moreover, he gives a proof (but not the first proof) that $\theta_F(U, V) = \theta_F(U^\perp, V^\perp)$, which gives further reason to prefer θ_F over θ_1 .

Another proper angle distance is the maximum angle θ_k , first considered by Krein, Krasnoselski, and Milman [12]. It was used by Asimov [1] for his “Grand Tour”, a method for visualizing high dimensional data by projecting to various two-dimensional subspaces and showing these projections sequentially to a human. θ_k was also considered by Conway, Hardin, and Sloane [6] in their paper on packing subspaces in Grassmannians.

For any angle distance d and $\alpha > 0$, we give an upper bound on $N_\alpha^d(k, n)$ on the order of n^{2k} , extending Gerzon’s bound in eq. (1). In the case $d = \theta_1$ and $k = 2$, this improves Blokhuis’ bound in eq. (3). The proof is based on the polynomial method, which was also the main tool in [4].

Theorem 1. *Let $k, n \in \mathbb{N}$ with $k \leq n$, let d be an angle distance on $\text{Gr}(k, n)$ and let $\alpha > 0$. Then*

$$N_\alpha^d(k, n) \leq \binom{\binom{n+1}{2} + k - 1}{k}.$$

Proof. Let $\{M_1, \dots, M_m\} \subseteq \text{Gr}(k, n)$ be an equidistant set such that $d(M_i, M_j) = \alpha$ for all $i \neq j$ and let $\lambda = (\cos \alpha)^2$. Let $S = \{X \in \mathbb{R}^{n \times n} : X^\top = X\}$ be the set of all symmetric $n \times n$ matrices and define functions $f_1, \dots, f_m : S \rightarrow \mathbb{R}$ by

$$f_i(X) = \det \left(M_i^\top X M_i - \frac{\lambda \text{tr}(X)}{k} I \right).$$

Now observe that for any $i \neq j$, since $\alpha = d(M_i, M_j)$ is a principle angle between M_i and M_j , we have that λ is an eigenvalue of $M_j^\top M_i M_i^\top M_j$. Moreover, we observe that $\text{tr}(M_j M_j^\top) = k$ for all j , so we conclude

$$f_i(M_j M_j^\top) = \begin{cases} (1 - \lambda)^k & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

Moreover, note that $\lambda \neq 1$ since $\alpha \neq 0$. It therefore follows that f_1, \dots, f_m are linearly independent. Indeed, if $\sum_{i=1}^m c_i f_i = 0$ for some $c_1, \dots, c_m \in \mathbb{R}$, then for all j we have $0 = \sum_{i=1}^m c_i f_i(M_j M_j^\top) = c_j (1 - \lambda)^k$, which implies $c_j = 0$.

Thus it suffices to show that f_1, \dots, f_m live in a space of dimension $\binom{n+1}{k}^{+k-1}$. To that end, recall that a multivariable polynomial $f : \mathbb{R}^t \rightarrow \mathbb{R}$ is called homogeneous of degree r if it is a linear combination of monomials of degree r , and that the linear space of such polynomials has dimension $\binom{t+r-1}{r}$. Now observe that for any i and $X \in S$, every entry of $M_i^\top X M_i - \frac{\lambda \text{tr}(X)}{k} I$ is a homogeneous polynomial of degree 1 in the variables $\{X_{a,b} : 1 \leq a \leq b \leq n\}$. It follows from the definition of the determinant that $f_i(X)$ is a homogeneous polynomial of degree k in the variables $\{X_{a,b} : 1 \leq a \leq b \leq n\}$. Since there are $\binom{n+1}{2}$ such variables, the space of all homogeneous polynomials of degree k in these variables has dimension $\binom{n+1}{k}^{+k-1}$, completing the proof. \square

To obtain lower bounds for this problem, it is natural to start with a construction of many equiangular lines and then try to combine them to make k -dimensional subspaces. In the following, we make use of the Frobenius inner product $\langle A, B \rangle = \text{tr}(A^\top B)$ for $n \times n$ real-valued matrices.

Theorem 2. *For any $k, n \in \mathbb{N}$ with $k \leq n$, there exists a set $H \subseteq \text{Gr}(k, kn)$ with $|H| = N(n)^k$ and $\alpha \in (0, \pi/2)$ such that for all $U, V \in H$, the principle angles between U and V all lie in the set $\{0, \alpha\}$.*

Proof. Let $\alpha \in (0, \pi/2)$ be such that there exists a set of equiangular lines $C \subseteq \text{Gr}(1, n)$ with $|C| = N(n)$ and common angle α . Thinking of C now as a set of unit vectors, we have that $\langle u, v \rangle^2 = \cos^2 \alpha$ for all $u \neq v \in C$.

Now let e_1, \dots, e_k be the standard basis in \mathbb{R}^k , and observe that for all $u, v \in C$, we have

$$\langle e_i u^\top, e_j v^\top \rangle = \text{tr}(u e_i^\top e_j v^\top) = (e_i^\top e_j)(u^\top v) = \begin{cases} \langle u, v \rangle & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

Thus if we let $v_1, \dots, v_k \in C$ then $e_1 v_1^\top, \dots, e_k v_k^\top$ can be viewed as orthonormal vectors in \mathbb{R}^{kn} and hence define a subspace W_{v_1, \dots, v_k} in $\text{Gr}(k, kn)$. Furthermore, for all $u_1, \dots, u_k, v_1, \dots, v_k \in C$ if we let $U = W_{u_1, \dots, u_k}$ and $V = W_{v_1, \dots, v_k}$, we obtain

$$(U^\top V)_{i,j} = \langle e_i u_i^\top, e_j v_j^\top \rangle = \begin{cases} \langle u_i, v_j \rangle & \text{if } i = j \\ 0 & \text{if } i \neq j, \end{cases}$$

and hence

$$(V^\top U U^\top V)_{i,j} = \langle e_i u_i^\top, e_j v_j^\top \rangle = \begin{cases} \langle u_i, v_j \rangle^2 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

Thus the eigenvalues of $V^\top U U^\top V$ lie in the set $\{1, \cos(\alpha)^2\}$ and so the principle angles between U and V lie in the set $\{0, \alpha\}$. Letting $H = \{W_{v_1, \dots, v_k} : v_1, \dots, v_k \in C\}$ and observing that $|H| = N(n)^k$ completes the proof. \square

We show that the construction above is equiangular for any proper angle distance d , and hence obtain the following corollary.

Corollary 1. *Let d be a proper angle distance and let $k \in \mathbb{N}$ be fixed. Then*

$$N^d(k, n) = \Theta(n^{2k}) \text{ as } n \rightarrow \infty.$$

Proof. Theorem 1 immediately gives the upper bound $N^d(k, n) \leq O(n^{2k})$. For the lower bound, let $\alpha \in (0, \pi/2)$ and $H \subset \text{Gr}(k, kn)$ be given by Theorem 2. Observe that for all $U \neq V \in H$, the principle angles between U and V cannot all be 0, and thus $\theta_F(U, V) = \theta_k(U, V) = \alpha$. Moreover, observe that since d is a proper angle distance, we have $\theta_F \leq d \leq \theta_k$. Thus $d(U, V) = \alpha$ for all $U \neq V \in H$. De Caen's bound eq. (2) implies that $N(n) \geq \Omega(n^2)$ and so we obtain

$$N^d(k, kn) \geq |H| = N(n)^k \geq \Omega(n^{2k}).$$

Thus we conclude $N^d(k, n) \geq \Omega(n^{2k})$. \square

3 Concluding remarks

In section 2 we give an upper bound on $N_\alpha^d(k, n)$ of the order n^{2k} for any angle distance d and $\alpha > 0$, but are only able to give a corresponding lower bound when d is a proper angle distance. It would therefore be interesting to give lower bound constructions (with common angle $\alpha > 0$) on the order of n^{2k} for angle distances that are not proper, in particular for the minimum angle θ_1 . Moreover, if $n \gg k \rightarrow \infty$ then even for proper angle distances d , Corollary 1 still leaves open the correct asymptotic dependence of $N^d(k, n)$ on k .

Another approach to generalizing equiangular lines is, given a set $H \subseteq \text{Gr}(k, n)$, to require that H is equiangular with respect to θ_i for all $1 \leq i \leq k$. If we further require that $\theta_1 = \dots = \theta_k$, we arrive at the notion of *equi-isoclinic* subspaces. Equivalently, a family of subspaces $H \subseteq \text{Gr}(k, n)$ is equi-isoclinic if there exists $\lambda \in [0, 1)$ such that $V^\top U U^\top V = \lambda I$ for all $U \neq V \in H$. Lemmens and Seidel [14] defined and studied $v(k, n)$, the maximum number of k -dimensional equi-isoclinic subspaces in \mathbb{R}^n . They gave a construction based on equiangular lines showing that $v(1, n) \leq v(k, kn)$ and generalized Gerzon's bound in eq. (1), obtaining $v(k, n) \leq \binom{n+1}{2} - \binom{k+1}{2} + 1$. Note that for $n \gg k \rightarrow \infty$, these bounds together with the fact that $v(1, n) = N(n) \geq \Omega(n^2)$ show that

$$\Omega\left(\frac{n^2}{k^2}\right) \leq v(k, n) \leq O(n^2).$$

It would be interesting to close this gap and determine the correct asymptotic dependence of $v(k, n)$ on k .

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